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Minimum Heat Input Optimal Skip Entry into Venus Atmosphere

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We present the optimal skip entry into Venus atmosphere of a space vehicle with minimum total heat input. The motion of the lifting body vehicle follows a plane flight path, and the variation of the Venus density with altitude has an exponential law. The variational problem is a Lagrange problem, which becomes a bilocal problem and can be solved numerically with the transition matrix method. The numerical applications have practical interest for the vehicle's evolutions into Venus atmosphere.

Nomenclature

C_D	= drag coefficient
C'_f	= skin-friction coefficient
C_L	= lift coefficient
D	= drag force $[D = (\rho/2)SV^2C_D]$
$\mathrm{d}q_m/\mathrm{d}t$	= mean flux, $(C'_f/4)\rho V^3$
g	= gravity acceleration
L	= lift force $[L = (\rho/2)SV^2C_L]$
l	= length of reference
m	= vehicle's mass
Q	= $S_V \int_{t_i}^{t_f} (\mathrm{d}q_m/\mathrm{d}t)\,\mathrm{d}t$
Re	= Reynolds number, $(\rho V l/\mu)$
R_0	= planet radius
r	= vehicle vector radius
S	= surface of reference
S_V	= vehicle surface area
V	= vehicle velocity
x	= distance on horizontal
z	= altitude
β	= constant in definition of the density variation with the altitude
θ	= flight-path angle
μ	= dynamic viscosity
ρ	= density, $\rho_0 e^{-\beta z}$

Introduction

RESEARCH on the Venus high atmosphere, as well as on its relief, may be carried out by an instrumented vehicle for measurements and imagery. This instrumented vehicle, achieved as a lifting body with thermal protection, would descend into the atmosphere from a circular orbit, complete an evolution on a skip bidimensional flight path, and return on the initial orbit.

As we examine the atmospheric phase, we note that the problem of optimization of the space vehicle's entry into the Venus atmosphere, in its various aspects, is of primary concern. One aspect of atmosphere entry optimization is minimization of the total heat input, which is of practical interest.

This work deals with the determination of the optimal law motion parameters variation and the control variable, so that the vehicle completes an evolution on the skip flight path with minimum heating.

We consider the convection heat transfer to be dominant and the variation of the skin-friction coefficient with the Reynolds number to have no significant influence on the motion parameters.[1] The variation of density with the altitude is considered according to an exponential law established based on the measurements performed by the station Venus-8.

Motion Equations

Considering the θ flight-path angle positive at the descent and negative at the ascent, with the approximation $r = R_0 + z \approx R_0$, the motion equations for two-dimensional entry can be written as follows:

$$m\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{\rho}{2}SV^2C_D + mg\sin\theta$$
$$-mV\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\rho}{2}SV^2C_L - m\left(g - \frac{V^2}{R_0}\right)\cos\theta \tag{1}$$

to which are added the cinematic equations

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -V\sin\theta, \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = V\cos\theta \tag{2}$$

Supposing the vehicle entry with small angles and with lift force other than zero, and neglecting the small terms, the system of equations that governs the motion can be kept with the Eggers assumption in the form

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{SC_D}{2m}\rho V^2, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = -V\sin\theta$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{SC_L}{2m}\rho V, \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = V\cos\theta \tag{3}$$

Variational Problem

For skip entry study, the first three equations of Eq. (3) will be used rewritten as

$$\phi_V = \frac{\mathrm{d}V}{\mathrm{d}t} + \frac{SC_D}{2m}\rho V^2 = 0, \qquad \phi_\theta = \frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{SC_L}{2m}\rho V = 0$$
$$\phi_z = \frac{\mathrm{d}z}{\mathrm{d}t} + V\sin\theta = 0 \tag{4}$$

Because the skip entry is shallow, adopting the model of laminar heat transfer and supposing that the variation of the skin-friction coefficient with Reynolds number does not significantly influence the motion parameters,[1] the vehicle total heat input can be expressed as

$$Q = \frac{S_V C'_f}{4}\int_{t_i}^{t_f} \rho V^3\,\mathrm{d}t \tag{5}$$

The variational problem is to find the optimal variational laws of the motion parameters and control variable C_L so that the functional equation (5) becomes minimal with Eq. (4) conditions.

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Taking into account Eqs. (4) and (5) and using the Lagrange multipliers λ_V, λ_θ, and λ_z, the function

$$H = \rho V^3 + \lambda_V \phi_V + \lambda_\theta \phi_\theta + \lambda_z \phi_z \tag{6}$$

is formed. Considering the exponential law for the density

$$\rho = \rho_0 e^{-\beta z}$$

with vehicle parabolic drag

$$C_D = C_{D0} + kC_L^2$$

Euler's equations

$$\frac{\partial H}{\partial V} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial H}{\partial(\mathrm{d}V/\mathrm{d}t)} = 0, \qquad \frac{\partial H}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial H}{\partial(\mathrm{d}\theta/\mathrm{d}t)} = 0$$
$$\frac{\partial H}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial H}{\partial(\mathrm{d}z/\mathrm{d}t)} = 0, \qquad \frac{\partial H}{\partial C_L} = 0 \tag{7}$$

lead to the differential equations system

$$\frac{\mathrm{d}\lambda_V}{\mathrm{d}t} = 3\rho_0 e^{-\beta z} V^2 + \lambda_V \frac{\rho_0 S}{m}\left(C_{D0} + kC_L^2\right)$$
$$\times e^{-\beta z} V + \lambda_\theta \frac{\rho_0 S}{2m} C_L e^{-\beta z} + \lambda_z \sin\theta$$
$$\frac{\mathrm{d}\lambda_\theta}{\mathrm{d}t} = \lambda_z V \cos\theta \tag{8}$$
$$\frac{\mathrm{d}\lambda_z}{\mathrm{d}t} = -\beta\rho_0 e^{-\beta z} V^3 - \lambda_V \beta \frac{\rho_0 S}{2m}\left(C_{D0} + kC_L^2\right)$$
$$\times e^{-\beta z} V^2 - \lambda_\theta \beta \frac{\rho_0 S}{2m} C_L e^{-\beta z} V$$

and the algebraic equation

$$C_L = -\frac{\lambda_\theta}{2\lambda_V k V} \tag{9}$$

Taking into account expression (9) for C_L and using the notation $a = (\rho_0 S)/2m$, system (8) becomes

$$\frac{\mathrm{d}\lambda_V}{\mathrm{d}t} = 3\rho_0 e^{-\beta z} V^2 + 2\lambda_V a e^{-\beta z} V C_{D0} + \lambda_z \sin\theta$$
$$\frac{\mathrm{d}\lambda_\theta}{\mathrm{d}t} = \lambda_z V \cos\theta \tag{10}$$
$$\frac{\mathrm{d}\lambda_z}{\mathrm{d}t} = -\beta e^{-\beta z}\left[\rho_0 V^3 + aV\left(\lambda_V V C_{D0} + \frac{\lambda_\theta}{2} C_L\right)\right]$$

Extremum Nature Analysis

For the extremum nature analysis let us use, for simplification, the state variables and their derivatives in time as follows:

$$V = x_1, \qquad \frac{\mathrm{d}V}{\mathrm{d}t} - \dot{x}_1; \qquad \theta = x_2, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{x}_2$$
$$z = x_3, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = \dot{x}_3 \tag{11}$$

The Legendre conditions for a weak minimum can be written as

$$\frac{\partial^2 H}{\partial \dot{x}_1^2} \geq 0$$
$$\cdots\cdots\cdots\cdots$$
$$\begin{vmatrix} \frac{\partial^2 H}{\partial \dot{x}_1^2} & \frac{\partial^2 H}{\partial \dot{x}_1 \partial \dot{x}_2} & \frac{\partial^2 H}{\partial \dot{x}_1 \partial \dot{x}_3} \\ \frac{\partial^2 H}{\partial \dot{x}_2 \partial \dot{x}_1} & \frac{\partial^2 H}{\partial \dot{x}_2^2} & \frac{\partial^2 H}{\partial \dot{x}_2 \partial \dot{x}_3} \\ \frac{\partial^2 H}{\partial \dot{x}_3 \partial \dot{x}_1} & \frac{\partial^2 H}{\partial \dot{x}_3 \partial \dot{x}_2} & \frac{\partial^2 H}{\partial \dot{x}_3^2} \end{vmatrix} \geq 0 \tag{12}$$

The function H can be seen from Eq. (6) to have a linear dependence with $\dot{x}_1$, $\dot{x}_2$, and $\dot{x}_3$ so that the Legendre conditions for a weak minimum are fulfilled. To demonstrate the Weierstrass condition for a strong minimum, let us consider the function

$$E = H\left(x, u^*, \lambda, \frac{\mathrm{d}x^*}{\mathrm{d}t}\right) - H\left(x, u, \lambda, \frac{\mathrm{d}x}{\mathrm{d}t}\right)$$
$$-\left(\frac{\mathrm{d}x^*}{\mathrm{d}t} - \frac{\mathrm{d}x}{\mathrm{d}t}\right)\left[\frac{\partial H(x, u, \lambda, \mathrm{d}x/\mathrm{d}t)}{\partial(\mathrm{d}x/\mathrm{d}t)}\right]^T \tag{13}$$

where values with asterisks represent the variables on the extremal and where

$$\lambda = (\lambda_V, \lambda_\theta, \lambda_z), \qquad u = C_L$$
$$x = (x_1, x_2, x_3) = (V, \theta, z) \tag{14}$$

After calculation we have

$$E = \frac{\lambda_V S k V^2}{2m} - \left(C_L^* - C_L\right)^2 \tag{15}$$

The numerical calculi show that $\lambda_V > 0$ and $\lambda_\theta < 0$, and for this reason

$$E > 0 \tag{16}$$

The Weierstrass condition for a strong minimum is satisfied, and in addition, from Eq. (9) we have the result that $C_L > 0$, this condition being necessary for the skip entry. The variational problem so formulated has sense and is indeed a problem of minimum.

Calculus of the Optimal Variation Laws of the Motion Parameters

To resolve the presented problem, the six differential equations (4) and (10) with six unknown functions V, θ, z, λ_V, λ_θ, and λ_z must be solved. The state variables and the independent variable are considered to be given in the initial moment:

$$V_i = V(t_i), \qquad \theta_i = \theta(t_i), \qquad z_i = z(t_i), \qquad t_i = 0 \tag{17}$$

with λ_{V_i}, λ_{θ_i}, and λ_{z_i} unspecified.

At the end of the evolution, the final condition is

$$z_f = z_i \tag{18}$$

with the variables V_f, θ_f, and t_f unspecified. Usually V_f is prescribed because this simplifies the calculus, but without sufficient specific information for Venus it is difficult to prescribe this value. As for the variables λ_{V_f}, λ_{θ_f}, and λ_{z_f}, these can result from the transversality condition written in general form with notations from Eq. (11):

$$\left[\left(H - \sum_{j=1}^{3} \dot{x}_j \frac{\partial H}{\partial \dot{x}_j}\right)\delta t + \sum_{j=1}^{3} \frac{\partial H}{\partial \dot{x}_j}\delta x_j\right]_i^f = 0 \tag{19}$$

Based on this and because $\delta z = 0$ (z_f is known), we have

$$\lambda_{vf} = \lambda_{\theta f} = 0$$
$$\left[H - \lambda_v \frac{\mathrm{d}V}{\mathrm{d}t} - \lambda_\theta \frac{\mathrm{d}\theta}{\mathrm{d}t} - \lambda_z \frac{\mathrm{d}z}{\mathrm{d}t}\right]^f = 0 \tag{19'}$$

or

$$\lambda_{vf} = \lambda_{\theta f} = 0, \qquad \lambda_{zf} = \frac{\rho_0 e^{-\beta z_f} V_f^2}{\sin\theta_f}$$

The problem so formulated is a bilocal problem[2,3] and presents the characteristic difficulty for this type of problem: The solution of the differential equations (10) cannot be obtained by a simple integration because the initial values are not known. The problem can be numerically resolved with the transition matrix method.[2] We use the notation

$$[\Lambda] = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \tag{20}$$

for the column vector $[\lambda_V \;\; \lambda_\theta \;\; \lambda_z]^T$, where $[\Lambda_i]$ represents the initial values vector and $[\Lambda_f]$ the final values vector. A variation $[\delta\Lambda_i]^T = (\delta\lambda_{1i}, \delta\lambda_{2i}, \delta\lambda_{3i})$ of the initial values vector gives a variation $[\delta\Lambda_f]^T = (\delta\lambda_{1f}, \delta\lambda_{2f}, \delta\lambda_{3f})$ of the final values vector. The transition matrix $[T]$ is defined as

$$[\delta\Lambda_f] = [T][\delta\Lambda_i] \tag{21}$$

and, inversely,

$$[\delta\Lambda_i] = [T]^{-1}[\delta\Lambda_f] \tag{22}$$

The transition matrix can be calculated with $[\Lambda]$ equations in perturbation that are simultaneously integrated with the optimality equations (4), (9), and (10). The equations in perturbation are obtained from Eqs. (10) by differentiation. If system (10) is written

$$\frac{\mathrm{d}\lambda_k}{\mathrm{d}t} = f_k(V, \theta, z, \lambda_1, \lambda_2, \lambda_3, C_L), \qquad k = 1, 2, 3 \tag{23}$$

then the equations in perturbation are

$$\frac{\mathrm{d}(\delta\lambda_k)}{\mathrm{d}t} = \sum_{j=1}^{3} \frac{\partial f_k}{\partial \lambda_j}\delta\lambda_j + \frac{\partial f_k}{\partial C_L}\delta C_L$$

or in another form,

$$\frac{\mathrm{d}(\delta\lambda_V)}{\mathrm{d}t} = 2ae^{-\beta z} V C_{D0}\delta\lambda_V + \sin\theta\delta\lambda_z$$

$$\frac{\mathrm{d}(\delta\lambda_\theta)}{\mathrm{d}t} = V\cos\theta\delta\lambda_z \tag{24}$$

$$\frac{\mathrm{d}(\delta\lambda_z)}{\mathrm{d}t} = \beta a e^{-\beta z} V\left[V C_{D0}\delta\lambda_V + C_L\left(\delta\lambda_\theta - \frac{1}{2}\frac{\lambda_\theta}{\lambda_V}\delta\lambda_V\right)\right]$$

To determine the matrix of transition, let us first remark that a variation as $[\delta\lambda_{1i} \;\; 0 \;\; 0]^T$ produces a variation $[\delta\lambda_{1f} \;\; \delta\lambda_{2f} \;\; \delta\lambda_{3f}]^T$ so that

$$t_{11} = \frac{\delta\lambda_{1f}}{\delta\lambda_{1i}}, \qquad t_{21} = \frac{\delta\lambda_{2f}}{\delta\lambda_{1i}}, \qquad t_{31} = \frac{\delta\lambda_{3f}}{\delta\lambda_{1i}}$$

namely, the elements of column j from matrix $[T]$ represent exactly the variations $[\delta\Lambda_f]$ for an unitary variation of the variable λ_{ji}; the other variations of vector $[\Lambda_i]$ begin null. Integrating three times Eqs. (24), together with the optimality system and the initial conditions

$$\delta\lambda_{ji} = \delta_{jk}, \qquad j = 1, 2, 3, \qquad k = 1, 2, 3 \tag{25}$$

where δ_{jk} is the Kronecker delta, we obtain

$$\begin{bmatrix} t_{1k} \\ t_{2k} \\ t_{3k} \end{bmatrix} = \begin{bmatrix} \delta\lambda_{1f} \\ \delta\lambda_{2f} \\ \delta\lambda_{3f} \end{bmatrix} \tag{26}$$

Because vector $[\Lambda_f]$ is known, the following algorithm is used:

1) With $[\Lambda_i]_{\mathrm{old}}$ values, which initially had been arbitrarily chosen, the matrix of transition is calculated as shown earlier.

2) The matrix of transition is inverted.

3) Vector $[\delta\Lambda_f] = [\Lambda_{fc}] - [\Lambda_{fe}]$ is calculated, where $[\Lambda_{fc}]$ is the vector calculated by integration and $[\Lambda_{fe}]$ is the vector of accurate values.

For the calculus of $\lambda_{3f} = \lambda_{zf}$, it is taken into account that λ_{1f} and λ_{2f} are not the accurate values and represent the calculated values, and as a result

$$\lambda_{zf} = \left[\frac{H - \lambda_1(\mathrm{d}V/\mathrm{d}t) - \lambda_2(\mathrm{d}\theta/\mathrm{d}t)}{\mathrm{d}z/\mathrm{d}t}\right]_f \tag{27}$$

4) In general, $[\delta\Lambda_f]$ is not sufficiently small; thus, step 5 must be restated.

5) Calculate $[\delta\Lambda_i]$ using Eqs. (22).

6) Reevaluate vector $[\Lambda_i]$ as

$$[\Lambda_i]_{\mathrm{new}} = [\Lambda_i]_{\mathrm{old}} - \varepsilon[T]^{-1}[\delta\Lambda_f] \tag{28}$$

where ε is a positive number less than one introduced to compensate the linear approximation that modifies its value from one iteration to the other, tending to zero.

Farther away, the iterative process was taken again, starting step 1 with the $[\Lambda_i]$ values so modified, the process continuing until $\varepsilon \to 0$.

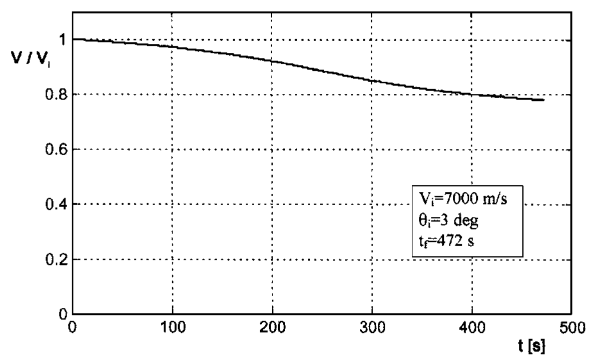

Fig. 1 Ratio of the velocity optimal variation and the initial velocity.

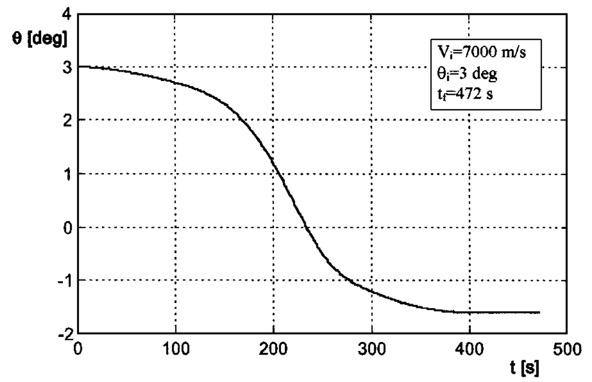

Fig. 2 Optimal variation of the flight-path angle θ.

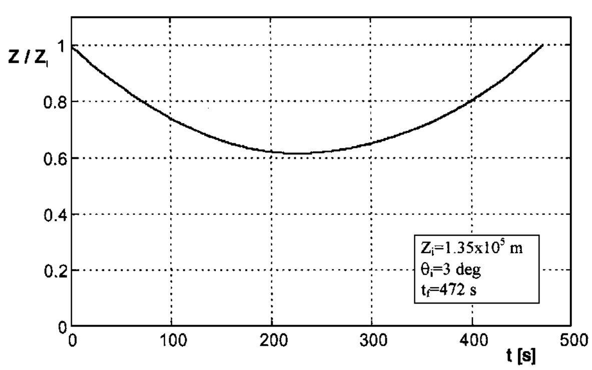

Fig. 3 Ratio of the altitude optimal variation and the initial altitude.

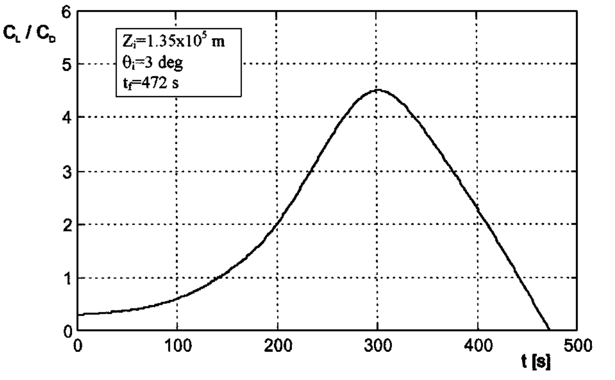

Fig. 4 Optimal variation of the C_L/C_D ratio.

Numerical Application

For better development of the calculations and to facilitate the choice of the initial values of the unspecified parameters, the following nondimensional parameters were used:

$$V^* = (V/V_i), \qquad \theta^* = \theta, \qquad z^* = (z/z_i)$$

With the data $V_i = 7000$ m/s, $k = 0.55$, $\theta_i = 3$ deg, $a = 0.03$ m^{-1}, $z_i = 1.35 \times 10^5$ m, $\beta = 1.493 \times 10^{-4}$ m^{-1}, $C_{D0} = 0.02$, and $\rho_0 = 6.645$ kg s^2 m^{-4}, the calculation was performed according to the described algorithm. The results are shown in Figs. 1–4. In Fig. 1, the ratio of the velocity optimal variation and initial velocity is shown, and we see an almost linear variation of this ratio. In Fig. 2 optimal variation of the flight-path angle θ is shown, and we see this angle change sign in the minimum altitude region. In Fig. 3 the ratio of the altitude optimal variation and the initial altitude is shown. In Fig. 4 the optimal ratio C_L/C_D is shown, and we see that the initial value of C_L/C_D is small.

Conclusions

Note that the velocity loss at the end of the evolution is greater compared with the current evolutions in the Earth atmosphere and the minimum altitude and the evolution time are also greater. This seems easy to explain due to the higher density of the Venus atmosphere. The C_L/C_D ratio, at its maximum value, should be inscribed with the achievable values concerning the lifting bodies.

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